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Sheet 10 for Effiziente Algorithmen (Winter 2025/26)

Hand In: *Until 2026-01-16 18:00, on ILIAS.*

Problem 1

20 + 20 points

In the lecture, the edges of a depth-first-search tree (DFS forest) were classified into the types *tree*, *forward*, *back*, and *cross*.

For a breadth-first-search tree the same types can be used. (Note that the names might be a bit less intuitive there!)

For a breadth-first search starting at a source vertex s, let $u.d$ denote the distance from s to u (in number of vertices).

a) Show that for breadth-first search on an *undirected* graph the following properties hold.
   - There are no *cross* and *forward* edges.
   - For every *tree* edge uv we have $v.d = u.d + 1$.
   - For every *back* edge uv we have $v.d = u.d$ or $v.d = u.d + 1$.

b) Show that for breadth-first search on a directed graph (digraph) the following properties hold.
   - There are no *forward* edges.
   - For every *tree* edge uv we have $v.d = u.d + 1$.
   - For every *back* edge uv we have $v.d = u.d$ or $v.d = u.d + 1$.
   - For every *cross* edge uv we have $v.d \leq u.d$.

Problem 2

20 + 30 points

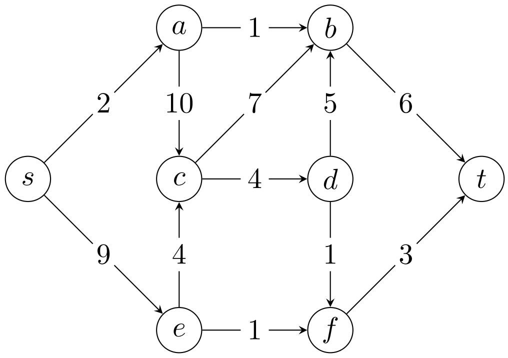
a) How can the number of strongly connected components in a directed graph change by inserting a new edge? Justify your answer.

b) We call a directed graph $G = (V, E)$ *semi-connected* if for every pair of vertices $u, v \in V$ there exists a path from u to v *or* a path from v to u. Develop an efficient algorithm that determines whether a directed graph G is semi-connected. Justify the correctness of your algorithm and analyze its running time.

Problem 3

40 + 10 points

Given the following flow network.



a) Determine the maximum flow f from s to t using the Ford–Fulkerson method. For each iteration, specify which s–t path is used, the amount of flow sent along that path, and the resulting residual graph.

b) Give a minimum s–t cut, i.e., sets S and T, such that the capacity of f across (S, T) equals the value of the maximum flow.

Problem 4

20 + 30 points

a) Prove or disprove the following statement:

Let (A, B) be a minimum s–t cut in a flow network G with source s and sink t, where every edge capacity is integral. If the flow network G' is obtained by increasing every edge capacity of G by exactly 1, then (A, B) is also a minimum s–t cut for G'.

b) Given a flow network $G = (V, E)$ where every edge $e \in E$ has capacity $c(e) = 1$, together with $s, t \in V$ and $k \in \mathbb{N}_{\geq 1}$, give an efficient algorithm that deletes k edges from G so that the maximum s–t flow becomes as small as possible. Justify the correctness of your solution.

Problem 5

30 + 20 points

After a natural disaster there are n injured people distributed across a region. Rescue teams must transport them to nearby hospitals. An injured person can only be taken to a hospital that can be reached within 30 minutes. Additionally, a hospita j can treat at most c_j injured persons.

You may assume that c_j is known and that for each person p_i the set of hospitals S_i that can reach p_i within 30 minutes is given.

a) Show (using a suitable and labeled example diagram) how the described problem can be modeled as a maximum-flow problem. Also indicate how, from the solution of your flow problem, one can read off whether there exists a feasible assignment such that each hospital j receives at most c_j injured persons and every injured person is assigned to a hospital reachable within 30 minutes.

b) In addition to all the above conditions, suppose we now have information about whether a hospital is a municipal hospital or a university hospital. The injured should now also be distributed so that $\lfloor n/2 \rfloor$ injured persons are placed in municipal hospitals and $\lceil n/2 \rceil$ injured persons are placed in university hospitals. Again, show (using a suitable and labeled example diagram) how this extension of the problem can be modeled as a maximum-flow problem.